General Relativity | Week 10

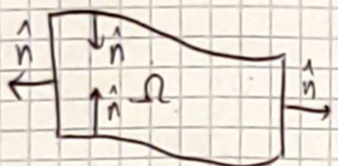
Last time: We saw that if ψ solves

$$\Box_g \psi = F \quad , \text{ setting } T_{\mu\nu}[\psi] = \partial_\mu\psi\, \partial_\nu\psi - \tfrac{1}{2}\partial^a\psi\,\partial_a\psi\, g_{\mu\nu}$$

then $\boxed{\operatorname{div} J^{(X)}[\psi] = F\cdot X(\psi) + T_{\mu\nu}[\psi]\, \underbrace{\nabla^\mu X^\nu}}$

(Where $J^{(X)}_\mu[\psi] = T_{\mu\nu}[\psi]\cdot X^\nu$) $\qquad \frac{1}{2}(\nabla^\mu X^\nu + \nabla^\nu X^\mu) = \frac{1}{2}(\mathcal{L}_X g)^{\mu\nu}$

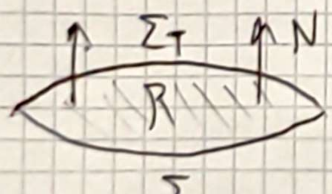
And: Divergence theorem: If $\Omega \subseteq (M,g)$ has only spacelike or timelike components for its boundary, then $\forall \omega \in \Gamma^*(M)$:



$$\int_\Omega \operatorname{div}\omega \; dvol_g = \int_{\partial\Omega} \omega(\hat{n})\; dvol_{\bar{g}}$$

We will use the above to get an energy-type inequality on general spacetimes.

Let R be a domain of a spacetime (M,g) that looks like a "lens", i.e.:



- $\overline{R}$ is compact
- $R = \bigcup_{\tau\in[0,T)} \Sigma_\tau$, Σ_τ: strictly spacelike $\rightarrow$ level set of function τ
- $\partial R = \Sigma_0 \cup \Sigma_T$
- $\exists$ timelike vector field N defined on a neighborhood of $\overline{R}$

Note that with these assumptions: (R,g) is globally hyperbolic, each Σ_τ is a Cauchy hypersurface.

The "time" function τ is smooth in the interior of R, but singular at the corners. It satisfies $\sup_R \frac{1}{|\nabla\tau|} \leq C$.

Let $\Box_g \psi = F$ on a neighborhood of $\bar{R}$.

Integrate over R the identity

$$\operatorname{div} J^{(N)} = F \cdot N(\psi) + T_{\mu\nu} \cdot \nabla^\mu N^\nu$$

$\Downarrow$ divergence theorem

$$① \quad \int_{\Sigma_T} J^{(N)}_\mu \hat{n}^\mu = \int_{\Sigma_0} J^{(N)}_\mu \hat{n}^\mu + \int_R T_{\mu\nu}[\psi] \nabla^\mu N^\nu \, d\mathrm{vol}_g \quad Ⓐ$$
$$+ \int_R F \cdot N\psi \, d\mathrm{vol}_g \quad Ⓑ$$

here: We fix $\hat{n}$ to be future directed unit normal to Σ_τ

Let us set $f(\tau) = \int_{\Sigma_\tau} J^{(N)}_\mu [\psi] \hat{n}^\mu$

Since $N, \hat{n}$ timelike and future directed:

In any local coordinate chart $f(\tau) \geq c \int_{\Sigma_\tau \cap \{\text{coordinate chart}\}} |\partial\psi|^2$

Because of compactness of $\bar{R}$: I can choose this constant uniformly.

For the terms Ⓐ, Ⓑ: Co-area formula: $\int_R h \, d\mathrm{vol}_g = \int_0^T \int_{\Sigma_\tau} h \cdot \frac{1}{|\nabla\tau|} d\mathrm{vol}_g d\tau$

Combining the co-area formula with the Cauchy-Schwarz inequality:

$$Ⓐ = \int_R T_{\mu\nu}[\psi] D^\mu N^\nu = \int_0^T \int_{\Sigma_\tau} T_{\mu\nu}[\psi] \cdot D^\mu N^\nu \frac{1}{|D\tau|} dvol_{\bar{g}} \, d\tau$$

In local coordinates: $|T_{\mu\nu}[\psi]| \lesssim |\partial\psi|^2$

$$\left| D^\mu N^\nu \cdot \frac{1}{|D\tau|} \right| \lesssim 1$$

$$\lesssim C \int_0^T f(\tau) \, d\tau$$

And $Ⓑ = \int_R F \cdot N\psi \leq \left(\int_R F^2\right)^{1/2} \left(\int_R (N\psi)^2\right)^{1/2}$

$$\leq C\left(\int_R F^2\right)^{1/2} \left(\int_0^T f(\tau) \, d\tau\right)^{1/2} \lesssim C\int_R F^2 + C\int_0^T f(\tau) d\tau$$

So ① $\Rightarrow$ $f(T) \leq f(0) + C\int_0^T f(\tau) d\tau + C\int_R F^2$

Gronwall's inequality

$$\Longrightarrow \quad \sup_{\tau \in [0,T]} f(\tau) \leq C' f(0) + C' \int_R F^2$$

C': Depends on R, g, N but not on ψ!

- In other words: <u>Energy inequality</u>

$$\sup_{\tau \in [0,T]} \|\partial\psi\|^2_{L^2(\Sigma_\tau)} \leq C' \left(\|\partial\psi\|^2_{L^2(\Sigma_0)} + \|F\|^2_{L^2(R)} \right)$$

We can also "improve" the above energy estimate as follows:

Let us set

$$\mathcal{E}^{(1)}[\psi](\tau) := \int_{\Sigma_\tau} \left(J^{(N)}_\mu[\psi]\hat{n}^\mu + \psi^2 \right) dvol_{\bar{g}}$$

$$\left(\text{So } \mathcal{E}^{(1)}[\psi](\tau) = \|\psi\|^2_{H^1(\Sigma_\tau)} + \|\hat{n}(\psi)\|^2_{L^2(\Sigma_\tau)} \right)$$

↑ only tangential derivatives to Σ_τ

then

$$\boxed{\sup_{\tau\in[0,T)} \mathcal{E}^{(1)}[\psi](\tau) \lesssim \mathcal{E}^{(1)}[\psi](0) + \|F\|^2_{L^2(R)}}$$

Proof: Given the previous energy identity:

It suffices to ~~show~~ estimate $\sup_{\tau\in[0,T)} \int_{\Sigma_\tau} \psi^2 \, dvol_{\bar{g}}$.

In any local coordinate system of the form $(\tau, x^1, \dots, x^n)$ on a domain U which is τ-invariant:

$$\int_{\Sigma_\tau \cap U} \psi^2 \, dvol_{\bar{g}} = \int_{\Sigma_0 \cap U} \psi^2 \, dvol_{\bar{g}} + \int_0^\tau \int_{\Sigma_s \cap U} \underbrace{\partial_\tau(\psi^2)}_{2\partial_\tau\psi\cdot\psi} \, dvol_{\bar{g}} \, ds$$

$$\leq \int_{\Sigma_0\cap U} \psi^2 + \int_0^\tau \Big(\underbrace{\int_{\Sigma_s \cap U} (\partial_\tau \psi)^2 \, dvol_{\bar{g}}}\Big) ds + \int_0^\tau \Big(\int_{\Sigma_s\cap U} \psi^2 \, dvol_{\bar{g}} \Big) ds$$

This term:

$$\lesssim \int_{\Sigma_s} J^{(N)}_\mu \hat{n}^\mu$$

(Near the corners of R: ∂_τ is "small")

So applying Gronwall ... ∎

Note: We proved the energy estimate for

$\Box_g \psi = F$, ψ real valued.

Exactly the same proof works for

$$\Box_g \psi^k + A^{ak}_{\lambda} \partial_a \psi^\lambda + B^k_\lambda \psi^\lambda = F^k$$

(i.e vector valued, "diagonal" principal symbol)

(just put the lower order terms on the right hand side and treat them like F)

But then: I can get higher order energy estimates:

If $\Box_g \psi + A \partial \psi + B\psi = F$

Then $\forall k$: If $V^{(k)} = \begin{pmatrix} \psi \\ \partial\psi \\ \vdots \\ \partial^k \psi \end{pmatrix}$ then

$$\Box_g V^{(k)} + \tilde{A}\, \partial V^{(k)} + \tilde{B}\, V^{(k)} = F^{(k)} = \begin{pmatrix} F \\ \partial F \\ \vdots \\ \partial^k F \end{pmatrix}$$

So applying the energy estimate for $V^{(k)}$:

We get $\boxed{\sup_{\tau \in [0,T)} \mathcal{E}^{(k)}[\psi](\tau) \lesssim \mathcal{E}^{(k)}[\psi](0) + \|F\|^2_{H^{k-1}(R)}}$

where

$$\mathcal{E}^{(k)}[\psi](\tau) = \sum_{|a|=0}^{k} \int_{\Sigma_\tau} |\partial^a \psi|^2, \qquad \partial^a = \partial_0^{a_0} \cdots \partial_n^{a_n}$$

if $a = (a_0, \dots, a_n)$

$(|a| = a_0 + \dots + a_n)$

(Can give a coordinate-independent formulation)

From this: <u>propagation</u> of <u>regularity</u> from the initial data.

In L^2-norms (this is not true in other L^p space, except in 1+1 dimensions)

Let us consider the initial value proplem on R:

$$\begin{cases} \Box_g \psi = F \\ \psi|_{\Sigma_0} = \psi_0, \quad \hat{n}(\psi)|_{\Sigma_0} = \psi_1 \end{cases} \qquad ①$$

Applying the energy estimate to the <u>difference</u> of two solutions:

- <u>Uniqueness</u>
- Continuous dependence of ψ on (ψ_0, ψ_1, F)

The map $(\psi_0, \psi_1, F) \longrightarrow \psi$

is continuous from $H^k(\Sigma_0) \times H^{k-1}(\Sigma_0) \times H^{k-1}(R) \longrightarrow L^\infty_t \mathcal{E}^{(k)}$

finite k-th energy

In order to establish <u>well-posedness</u> for ①: It remains to show the <u>existence</u> of solutions.

<u>Theorem</u>: ① has a solution ψ on R if (ψ_0, ψ_1, F) are smooth.

<u>Proof</u>: It suffices to show this for $(\psi_0, \psi_1) = (0, 0)$

(In general: Extend (ψ_0, ψ_1) to arbitrary function $\tilde{\psi}$, then $\psi - \tilde{\psi}$ solves $\Box(\psi - \tilde{\psi}) = F - \Box\tilde{\psi}$ with 0 initial data)

Assume w.l.o.g that $T = 1$.

Let $X = \{ \phi \in C^\infty(R) : \phi, d\phi|_{\Sigma_1} = 0 \}$

We want ψ to solve $\forall \phi \in X$:

$$\int_R \psi \cdot \Box_g \phi = \int_R F \cdot \phi \quad (W)$$

Weak ("distributional") formulation of the equation:

Makes sense for ψ just a "distribution".

We will show that a $\psi \in L^2(R)$ exists solving (W).

Let $Y = \{\Box_g \phi : \phi \in X\} \subseteq L^2(R)$

Define the functional $L : Y \to \mathbb{R}$

by $L[\Box_g \phi] = \int_R F \cdot \phi$

By the energy estimate: $\sup_\tau \|\phi\|_{L^2(\Sigma_\tau)} \lesssim \|\Box_g \phi\|_{L^2(R)}$

(solving backwards from Σ_1)

So $\|\phi\|_{L^2(R)} \lesssim \|\Box_g \phi\|_{L^2(R)}$

Hence: $\left|\int_R F \cdot \phi\right| \leq \|F\|_{L^2(R)} \|\phi\|_{L^2(R)} \lesssim \|F\|_{L^2(R)} \|\Box_g \phi\|_{L^2(R)}$

$\Rightarrow \quad |L[\Box_g \phi]| \lesssim \|F\|_{L^2(R)} \|\Box_g \phi\|_{L^2(R)}$

So L is bounded on $(Y, \|\cdot\|_{L^2})$

$\Longrightarrow$ Hahn-Banach: $\exists$ extension L to $L^2(R)$ with the same norm

By the Riesz representation

$\exists\, \psi \in L^2(R)$ s.t $\mathcal{L}[z] = \int_R \psi \cdot z$

So ψ solves the weak form of the equation

$\square$

One can actually show that ψ is more regular, and thus solves the "strong" version of the equation

(By showing boundedness not in L^2, but in H^{-k}, $k>0$)